without any difficulty and equation (9) would be superfluous. If the two points are not too distant a value for c from (7) may be used as a first approximation and its correction determined by (9). The constant c being thus found, the length of the arc is found by the evaluation of an elliptic of the second species.

NOTE.—As it is contended that in the published answer to Prof. Hall's Query (see p. 94) the series which represents the value of u converges so slowly that the method is inconvenient, another answer is here submitted as given by CHAS. H. KUMMELL.

To find the most convenient way of computing the numerical value of the definite integral

$$I = \int_0^{\frac{\pi}{2}} d\varphi \sqrt{(\sin\varphi)}. \tag{1}$$

We have

$$\int_0^{\frac{\pi}{2}} d\varphi (\sin\varphi)^{2m-1}(\cos\varphi)^{2n-1} = \tfrac{1}{2}B(m, n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}. \tag{2}$$

Placing $m = \frac{3}{4}$ and $n = \frac{1}{2}$ we have

$$I = \int_0^{\frac{\pi}{2}} d\varphi \sqrt{(\sin\varphi)} = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{4})}. \tag{3}$$

But $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(\frac{3}{4}) = \dfrac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$ by the theorem: $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$,

$\Gamma(\frac{5}{4}) = \frac{1}{4}\Gamma(\frac{1}{4})$ " " " : $\Gamma(n+1) = n\Gamma(n)$;

therefore

$$I = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma(\frac{1}{4})^2}. \tag{4}$$

The circumference of the lemniscate of Bernoulli may be expressed in terms of $\Gamma(\frac{1}{4})$ as follows:

The polar equation of the lemniscate is $r^2 = a^2\cos 2\theta$. Denoting the circumference by P we have

$$\tfrac{1}{4}P = a\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{(\cos 2\theta)}}. \tag{5}$$

Placing $\theta = \frac{1}{4}\pi - \frac{1}{2}\varphi$ we have

$$\tfrac{1}{4}P = \frac{a}{2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{(\sin\varphi)}} = \tfrac{1}{4}aB(\tfrac{1}{4}, \tfrac{1}{2}) = \frac{a}{4}\cdot\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} \text{ [by (2)], or}$$

$$P = a\frac{\Gamma(\frac{1}{4})^2}{\sqrt{(2\pi)}}. \tag{6}$$

Combining this with (4) we obtain

$$I = \frac{2a\pi}{P}. \tag{7}$$

The proposed integral is therefore found by dividing the circumference of the lemniscate into the circumscribed circle.

To determine now P place in (5) $\sin\theta = \surd\frac{1}{2}\times\sin\psi$, then

$$\tfrac{1}{4}P = a\int_0^{\frac{\pi}{2}}\frac{d\psi}{\surd(2-\sin^2\psi)}. \tag{8}$$

Take the more general form

$$E_0 = \int_0^{\psi_0}\frac{d\psi}{\surd(a_0^2\cos^2\psi+b_0^2\sin^2\psi)}, \tag{9}$$

which reduces to (7) by placing $a_0 = \surd 2$, $b_0 = 1$ and $\psi_0 = \frac{1}{2}\pi$.

By Landen's substitution, let

$$\tan(\psi_1-\psi_0) = (b_0\div a_0)\tan\psi_0, \tag{10}$$

$$a_1 = \tfrac{1}{2}(a_0+b_0);\ b_1 = \surd(a_0b_0); \tag{11}$$

then

$$E_0 = \frac{1}{2}\int_0^{\psi_1}\frac{d\psi}{\surd(a_1^2\cos^2\psi+b_1^2\sin^2\psi)} = \tfrac{1}{2}E_1. \tag{12}$$

Similarly let

$$\tan(\psi_2-\psi_1) = (b_1\div a_1)\tan\psi_1, \tag{10'}$$

$$a_2 = \tfrac{1}{2}(a_1+b_1);\ b_2 = \surd(a_1b_1); \tag{11'}$$

hence

$$E_1 = \frac{1}{2}\int_0^{\psi_2}\frac{d\psi}{\surd(a_2^2\cos^2\psi+b_2^2\sin^2\psi)} = \tfrac{1}{2}E_2. \tag{12'}$$

Continuing this process until $a_n = b_n$ we have finally

$$E_n = \frac{\psi_n}{b_n};\ \therefore E_0 = \frac{\psi_n}{2^n b_n}. \tag{13}$$

If $\psi_0 = \frac{1}{2}\pi$ then $\psi_1 = \pi$, $\psi_2 = 2\pi \ldots\ldots \psi_n = 2^{n-1}\pi$; therefore

$$\tfrac{1}{4}P = \frac{a\pi}{2b_n}. \tag{14}$$

Substituting this into (7) we have

$I = b_n =$ arithmetic-geometric mean of $\surd 2$ and 1.

The very convenient computation may be arranged as follows:

$a_0 = 1.4142135624$	$\log b_0 = 0.0000000.000$
$b_0 = 1.0000000000$	$\log a_0 = 0.1505149.979$
$a_1 = 1.2071067812$	$\log b_1 = 0.0752574.989$
$b_1 = 1.1892071150$	$\log a_1 = 0.0817456.897$
$a_2 = 1.1981569481$	$\log b_2 = 0.0785015.943$
$b_2 = 1.1981235214$	$\log a_2 = 0.0785137.106$
$a_3 = 1.1981402347$	$\log b_3 = 0.0785076.525$
	$\log a_3 = 0.0785076.525$
	$\log b_4 = 0.0785076.525 = \log b_n.$

$$\therefore \int_0^{\frac{\pi}{2}} d\varphi\surd\overline{\sin\varphi} = 1.1981402347.$$